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VARIATIONAL FORMULATIONS OF CERTAIN PROBLEMS OF THE THEORY OF THE FLOW OF RIGID-PLASTIC MEDIA*

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The mathematical content of variational problems of the mechanics of rigid-plastic media reduces to minimizing convex functionals in non-reflexive spaces of solenoidal vector fields. Different formulations are presented in /1/. Problems are examined there in which discontinuous velocity fields occur. The initial functional is undetermined in such fields. In this connection, the problem is posed of constructing that extension of the original set of kinematically allowable velocity fields as would contain all their possible discontinuities allowed by the mechanics of rigidly plastic media, and of continuing the original functional into the extension obtained. This problem is solved in this paper for the case when the velocity field is given on the whole surface of the rigid-plastic body.

1. We consider the bounded domain $\Omega \subset R^n$ $(n = 2, 3)$ whose boundary Γ satisfies the Lipschitz condition. The classical variational problem on the stationary flow of a rigid-plastic medium is to seek the velocity field $u = (u_i)$ such that

$$J(u) = \min_V J(v) \tag{1.1}$$

Here

$$J(v) = \sqrt{2}k_* \int_\Omega |\varepsilon(v)|\, dx$$

$$|\varepsilon|^2 = \varepsilon_{ij}\varepsilon_{ij}, \quad 2\varepsilon_{ij}(v) = v_{i,j} + v_{j,i}$$

$$V = \{v \in D^2(\Omega):\ \operatorname{div} v = 0 \text{ in} \Omega,\ v = U \text{ on } \Gamma\}$$

$$D^r(\Omega) = \left\{v:\ \|v\|_r = \frac{1}{\sqrt{n}} \|\operatorname{div} v\|_{L^r(\Omega)} + \int_\Omega (|\varepsilon^D(v)| + |v|)\, dx < +\infty\right\}$$

$U = (U_i)$ is a velocity field given in Γ, and $\varepsilon^D(v)$ is the deviator of the tensor $\varepsilon(v)$, $r \geqslant 1$, $i, j = 1, 2, \ldots, n$.

It is known that the space $D^1(\Omega)$ is included continuously in the space of summable functions $L^{n/(n-1)}(\Omega)^n$ and $L^1(\Gamma)^n$ /1, 2/. We shall assume that the vector-function U is a trace of the solenoidal field $u_0 \in W_2^1(\Omega)^n$ on Γ.

In general, the variational problem (1.1) has no solution since the set of kinematically allowable velocity fields V does not contain velocity fields describing discontinuities of the sliding type. In this connection, it is meaningful to give its expanded formulation by which the following requirements will be imposed:

1) The description of the expanded set of kinematically allowable velocity fields V_+ and the expanded functional Φ defined on V_+ in terms of a function of the points:

2) The value of the functional Φ in velocity fields from V will equal the value of the functional J in these same fields;

3) Problem (1.1) and its expansion have identical dual problems;

4) The magnitude of the lower exact face of the expanded problem equals the magnitude of the lower exact face of problem (1.1);

5) The expanded problem has a solution.

The third requirement is explained by the fact that the dual problem to (1.1) is always solvable and has an explicit mechanical meaning. Obviously, ant variational expansion of problem (1.1) must necessarily satisfy these five requirements.

2. We will now describe the set V_+ and the functional Φ. We will examine the space and set of symmetric tensor functions

$$\Sigma = \{\tau = (\tau_{ij}) : \tau_{ii} \in L^2(\Omega),\ \tau^D \in L^\infty(\Omega)^{n\times n}\}$$

$$\Sigma_n = \{\tau \in \Sigma : \operatorname{div} \tau = (\tau_{ij,j}) \in L^n(\Omega)^n\}$$

$$K = \{\tau \in \Sigma : |\tau^D(x)| \leqslant \sqrt{2}k_* \text{ for almost all } x \in \Omega\}$$

$$K_0 = \{\tau \in K : \tau_{ii}(x) = 0 \text{ for almost all } x \in \Omega\}$$

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The expansion of the set V has the form

$$V_+ = \{u \in L^{n/(n-1)}(\Omega)^n : \operatorname{div} u = 0 \text{ in } \Omega, \sup_{\tau \in K \cap \Sigma_n} |(\varepsilon(u_0), \tau) - (u - u_0, \operatorname{div} \tau)| < +\infty\}$$

Here

$$(\varepsilon(u), \tau) = \int_\Omega \varepsilon_{ij}(u)\tau_{ij}\,dx, \quad (u, \operatorname{div}\tau) = \int_\Omega u_i\tau_{ij,j}\,dx$$

We note that the set V_+ contains all possible velocity field discontinuities both within the domain Ω and on its boundary Γ. The very definition of this set depends on the boundary conditions in problem (1.1), which are thereby taken into account insofar as this is possible for the admissibility of the discontinuities. In particular, it can be shown that the projection of the velocity vector on the normal direction to the surface Γ satisfies the boundary condition in a certain sense.

We define the functional $\Phi: V_+ \to R^1$ as follows:

$$\Phi(v) = \sup_{\tau \in K \cap \Sigma_n} (\varepsilon(u_0), \tau) - (v - u_0, \operatorname{div}\tau)$$

Now the expansion of problem (1.1) consists of seeking the velocity field $u \in V_+$ such that

$$\Phi(u) = \min\nolimits_{V_+} \Phi(v) \tag{2.1}$$

We now formulate the fundamental assertions.

Theorem 1. The following relationships hold:

$$V \subset V_+ \tag{2.2}$$

$$\Phi(v) = J(v), \ \forall v \in V \tag{2.3}$$

Theorem 2. The variational problem (2.1) has at least one solution $u \in V_+$ such that

$$\Phi(u) = \min\nolimits_{V_+} \Phi(v) = \inf\nolimits_V J(v) \tag{2.4}$$

and a symmetric tensor function $\sigma \in K \cap \Sigma_n$ exists where

$$\operatorname{div} \sigma = 0 \text{ in } \Omega \ \text{(in the sense of the distributions)} \tag{2.5}$$

$$(\varepsilon(u_0), \tau - \sigma) - (u - u_0, \operatorname{div}(\tau - \sigma)) \leqslant 0, \ \forall \tau \in K \cap \Sigma_n \tag{2.6}$$

The assertions in Theorems 1 and 2 correspond to the five requirements presented in Sect.1.

The variational inequality (2.6) expresses the necessary condition for an extremum of the functional Φ. All the information about the solution is contained therein, in particular, about the nature and parameters of the discontinuity. If there is no discontinuity, i.e., $u \in V$, then it is equivalent to the Drucker postulate written in integral form.

3. We will now prove the theorems. It can be shown that the vector-functions from. $C_0^\infty(\Omega)^n$ are compact in the space

$$D_0^2(\Omega) = \{v \in D^2(\Omega): \ v = 0 \ \text{on} \ \Gamma\}$$

Since $u - u_0 \in D_0^2(\Omega)$, then by using the definition of the generalized divergence of a symmetric tensor τ, we obtain the following identity:

$$(\varepsilon(u), \tau) = (\varepsilon(u_0), \tau) - (u - u_0, \operatorname{div}\tau), \ \forall \tau \in \Sigma_n \tag{3.1}$$

Taking into account that the vector field u is solenoidal, and also the equality

$$J(u) = \sup_{\tau \in K \cap \Sigma_n} (\varepsilon(u), \tau), \ \forall u \in V \tag{3.2}$$

we deduce both assertions of Theorem 1 from the identity (3.1).

In proving Therem 2 we consider a family of convex and bounded sets of the Sobolev space $W_2^1(\Omega)^n$

$$V_m = \{v \in V: \ \|v\|_{W_2^1(\Omega)^n} \leqslant m\}, \quad m \geqslant \|u_0\|_{W_2^1(\Omega)^n}$$

From standard theorems on saddle points there results the existence of the functions $u_m \in V_m$, $\tau_m \in K_0$ such that

$$(\varepsilon(u_m), \tau) \leqslant (\varepsilon(u_m), \tau_m) \leqslant (\varepsilon(v), \tau_m), \ \forall \tau \in K_0, \ v \in V_m \tag{3.3}$$

Using the above-mentioned imbeddings and selecting a subsequence if necessary, we will have

$$u_m \to u \text{ weakly in } L^{n/(n-1)}(\Omega)^n;\ \tau_m \to \tau_*\ (*) - \text{weakly in } L^\infty(\Omega)^{n\times n}$$
$$\operatorname{div} u = 0 \text{ in } \Omega,\ \tau_* \in K_0$$

Psssing to the limit in inequality (3.3), we obtain

$$\inf_{V_*} J(v) \leqslant (\varepsilon(v), \tau_*),\ \forall v \in V_* = V \cap W_2^1(\Omega)^n$$

Applying reasoning customary for duality theory, we arrive at the relationships

$$\inf_{V_*} J(v) = \inf_{V_*} \sup_{K_*} (\varepsilon(v), \tau) = \inf_{V_*} (\varepsilon(v), \tau_*) = \max_{K_*} \inf_{V_*} (\varepsilon(v), \tau) = (\varepsilon(u_0), \tau_*) \tag{3.4}$$

from which an integral identity for τ_* follows

$$(\varepsilon(v), \tau_*) = 0,\ \forall v \in \{v \in W_2^1(\Omega)^n\colon \operatorname{div} v = 0 \text{ in } \Omega,\ v = 0 \text{ on } \Gamma\}$$

A function $t \in L^2(\Omega)$ exists such that /3/

$$(\varepsilon(v), \tau_*) + (t, \operatorname{div} v) = 0,\ \forall v \in \{v \in W_2^1(\Omega)^n\colon v = 0 \text{ on } \Gamma\}$$

and therefore

$$\sigma = (t\delta_{ij} + \tau_{*ij}) \in K \cap \Sigma_n,\ \operatorname{div} \sigma = 0 \text{ in } \Omega \tag{3.5}$$

We consider inequality (3.3) for the function $\tau \in K \cap \Sigma_n$ and we pass to the limit taking identity (3.1) into account. We consequently obtain

$$(\varepsilon(u_0), \tau) - (u - u_0, \operatorname{div} \tau) \leqslant (\varepsilon(u_0), \tau_*) = (\varepsilon(u_0), \sigma), \qquad \forall \tau \in K \cap \Sigma_n. \tag{3.6}$$

It follows from inequality (3.6) that $u \in V_+$.
The following inequality is then obtained from (3.4)–(3.6):

$$(\varepsilon(u_0), \tau) - (u - u_0, \operatorname{div} \tau) \leqslant (\varepsilon(u_0), \tau_*) = (\varepsilon(u_0), \sigma) = \\ (\varepsilon(u_0), \sigma) - (v - u_0, \operatorname{div} \sigma),\ \forall v \in V_+,\ \tau \in K \cap \Sigma_n \tag{3.7}$$

We deduce from (3.4) and (3.7)

$$\Phi(u) = \min_{V_+} \Phi(v) = \inf_{V_*} J(v) = (\varepsilon(u_0), \sigma)$$

The assertions of Theorem 2 result from (3.5), (3.7) and the inequalities

$$\inf_{V_+} \Phi(v) \leqslant \inf_V J(v) \leqslant \inf_{V_*} J(v)$$

4. We give an example of the problem in which there is no continuous solution. The purpose of the example is to show that the formal mathematical requirements imposed on the variational expansion of the initial problem results is such a formulation as is capable of extracting discontinuous solutions that are completely reasonable from a mechanical point of view.

We consider a concentric ring. We set $x_1 = \rho\cos\theta,\ x_2 = \rho\sin\theta$, where $\theta \in [0, 2\pi],\ \rho \in [R_1, R_2]$. Using polar coordinates we specify boundary conditions of the following kind:

$$u = (u_\rho, u_\theta);\ u_\rho = -U_0,\ u_\theta = 0 \text{ for } \rho = R_1\ \ u_\rho = -U_0/\alpha,\ u_\theta = U_* \text{ for } \rho = R_2$$

where U_0, U_* are positive constants, and $\alpha = R_2/R_1$.

Satisfying the boundary conditions for u_ρ at $\rho = R_1$ and $\rho = R_2$, for u_θ at $\rho = R_2$, and the equilibrium equations and relationships of the form

$$\varepsilon^D(u) = \lambda\sigma^D,\ \lambda \geqslant 0;\ (\sigma_\rho - \sigma_\theta)^2 + 4\sigma_{\rho\theta}^2 = 4k_*^2 \tag{4.1}$$

we obtain the stress tensor σ and the velocity field u dependent on the parameter c:

$$\sigma_{\rho\theta} = c\left(\frac{R_1}{\rho}\right)^2, \quad \sigma_\theta = \sigma_\rho - 2\sqrt{k_*^2 - c^2\left(\frac{R_1}{\rho}\right)^4} \tag{4.2}$$
$$\sigma_\rho = \sqrt{k_*^2 - c^2\left(\frac{R_1}{\rho}\right)^4} - k_* \ln\left(k_*\rho^2 + \sqrt{k_*^2\rho^4 - c^2R_1^4}\right) + p$$
$$\lambda = \frac{U_0R_1}{\rho^2\sqrt{k_*^2 - c^2\left(\frac{R_1}{\rho}\right)^4}}, \quad u_\rho = -U_0\frac{R_1}{\rho}$$
$$u_\theta = U_*\frac{\rho}{R_2} + \frac{U_0}{c}\cdot\frac{\rho}{R_1}\left[\sqrt{k_*^2 - c^2\left(\frac{R_1}{\rho}\right)^4} - \sqrt{k_*^2 - c^2\alpha^{-4}}\right]$$

Here p is an arbitrary constant and the parameter c varies between zero and k_*.

The parameter c is chosen from the condition $u_\theta = 0$ for $\rho = R_1$, which results in the following equation:

$$\sqrt{k_*^2 - c^2} + \sqrt{k_*^2 - c^2\alpha^{-4}} = (U_0/U_*)\,c\alpha\,(1 - \alpha^{-4}) \tag{4.3}$$

Equation (4.3) is solvable, and moreover uniquely, if the following inequality holds:

$$U_*/U_0 \leqslant \alpha(1-\alpha^{-4})^{1/2} \quad (4.4)$$

In this case the velocity field u defined by relations (4.1)—(4.3) will be a classical solution of problem (1.1).

If inequality (4.4) is not satisfied, we can set $c = k_*$ in (4.2). It then turns out that

$$u_\theta(R_1) = (U_0/\alpha)[U_*/U_0 - \alpha(1-\alpha^{-4})^{1/2}] > 0 \quad (4.5)$$

i.e., the boundary condition for $\rho = R_1$ is not satisfied for the function u_θ. Nevertheless, we show that the field u is a solution of problem (2.1). To do this, it is sufficient for the relationships (2.5) and (2.6) to be satisfied.

Indeed, it follows from (4.1) that

$$(\varepsilon(u), \tau - \sigma) \leqslant 0, \ \forall \tau \in K \quad (4.6)$$

By integration by parts we set up the equality

$$(\varepsilon(u), \tau - \sigma) = (\varepsilon(u_0), \tau - \sigma) - (u - u_0, \operatorname{div}(\tau - \sigma)) - \int\limits_{\rho = R_1} u_\theta(\tau_{\rho\theta} - k_*)d\Gamma,$$

for the smooth tensor functions $\tau \in K$.

Since $\tau \in K$ and inequality (4.5) is satisfied, then the contour integral is not positive in this last equality. Therefore

$$(\varepsilon(u), \tau - \sigma) \geqslant (\varepsilon(u_0), \tau - \sigma) - (u - u_0, \operatorname{div}(\tau - \sigma))$$

Now inequality (2.6) follows from the inequality (4.6). The value of the lower point of the face of problems (1.1) and (2.1) is evaluated from the formula

$$\Phi(u) = J(u) + k_* \int\limits_{\rho = R_1} u_\theta d\Gamma$$

where u is the solution of problem (2.1).

We finally note that by using (4.2) a solution can be constructed which will satisfy all the boundary conditions but the function u_θ has a jump within the domain. However, such a solution is intersected by problem (2.1) since it does not satisfy inequality (2.6).

5. We will briefly describe the possibility of the practical utilization of the expanded variational formulation.

Direct variational methods for the numerical solution of problem (1.1) require continuous approximations of the velocity fields. They can turn out to be inefficient in cases when problem (1.1) has no continuous solution.

When using the expanded formulation, the singularity assumed for the solution can be introduced directly into the basis functions of the method. For instance, in the finite element method it is sufficient to satisfy the continuity condition for the normal component of the velocity vector as it goes from one element to another. It is also sufficient to satisfy the boundary conditions for the velocity field component normal to the boundary. The surfaces of discontinuity therefore are approximated by the finite element faces. The real location of the discontinuities both within the domain and on its boundary is determined from the condition for a minimum of the functional Φ.

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